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ON THE RANK OF A MATRIX.

By W. H. Metzler.

1. It is a well-known theorem that: *If in a given matrix a certain m-rowed determinant is not zero, and all the $(m + 1)$-rowed determinants of which this m-rowed determinant is a first minor are zero, then all the $(m + 1)$-rowed determinants of the matrix are zero.*

The principal object of this paper is to give a proof of the following generalization* of the foregoing theorem and to give the corresponding generalizations for symmetric and skew-symmetric matrices:

If in a given matrix a certain m-rowed determinant is not zero, and all the $(m + h)$-rowed determinants of which this m-rowed determinant is a minor are zero, then all the $(m + h)$-rowed determinants of the matrix are zero.

Let

$$M \equiv \begin{vmatrix} (n \mid \underset{\alpha}{m}) \\ (n \mid \underset{\alpha}{m}) \end{vmatrix}^{\dagger} \neq 0, \qquad \text{and} \qquad \begin{vmatrix} (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{\beta}{h}) \\ (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{\gamma}{h}) \end{vmatrix} = 0,$$

for

$$\left.\begin{matrix} \beta \\ \gamma \end{matrix}\right\} = 1, 2, \cdots (n - m)_h.$$

On account of identical columns we obviously have

$$\begin{vmatrix} (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{\delta}{h + k}) \\ (n \mid \underset{\alpha}{m})(n \mid \underset{\alpha}{m} \mid \underset{i}{k})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{\gamma}{h}) \end{vmatrix} \equiv 0.$$

Expanding in terms of minors of order k and their complementaries we have

$$\sum_{j=1}^{m_k} \begin{vmatrix} (n \mid \underset{\alpha}{m} \mid \underset{j}{k}) \\ (n \mid \underset{\alpha}{m} \mid \underset{i}{k}) \end{vmatrix} \cdot \begin{vmatrix} (n \mid \underset{\alpha}{\bar{m}} \mid \underset{j}{k})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{\delta}{h + k}) \\ (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{\gamma}{h}) \end{vmatrix} = 0,$$

for $i = 1, 2, \cdots m_k$, and $k = 0, 1, \cdots m$.

The determinant of the coefficients‡ in this set of equations for each

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[*] This extension was proved by the writer in February, 1913, but before it was put in complete form for publication Cullis's book on Matrices and Determinoids appeared and contains the extension. While the theorem is no longer novel the method of proof here employed is of interest.

[†] Obviously no generality is lost in taking the m-rowed determinant coaxial.

[‡] By coefficient here is meant the first factor in each term.

value of k is the kth compound of M and is therefore not zero. Giving k successively the values from 1 to m we have

$$\begin{vmatrix} (n \mid \underset{\alpha}{\overline{m}} \mid \underset{j}{k})(\overline{n} \mid \underset{\alpha}{m} \mid h \underset{\delta}{+} k) \\ (n \mid \underset{\alpha}{m})(\overline{n} \mid \underset{\alpha}{m} \mid \underset{\gamma}{h}) \end{vmatrix} = 0, \tag{1}$$

for all values of j, δ, γ, k.

For the values 1, 2, $\cdots$ m of k we have the result that all minors of the matrix which have m columns and $(m - 1)$, $(m - 2)$, $\cdots$ 0, rows in common with M vanish.

Therefore

$$\begin{vmatrix} (n \mid m \underset{\epsilon}{+} h) \\ (n \mid \underset{\alpha}{m})(\overline{n} \mid \underset{\alpha}{m} \mid \underset{\gamma}{h}) \end{vmatrix} = 0, \tag{2}$$

for $\epsilon = 1, 2, \cdots n_{m-h}$.

LEMMA. If

$$\begin{vmatrix} (n \mid m \underset{\epsilon}{+} h) \\ (n \mid \underset{\alpha}{m})(\overline{n} \mid \underset{\alpha}{m} \mid \underset{\gamma}{h}) \end{vmatrix} = 0,$$

for all values of ϵ, γ, then

$$M' \equiv \begin{vmatrix} (n \mid m + \underset{\eta}{h} + k) \\ (n \mid \underset{\alpha}{m})(\overline{n} \mid \underset{\alpha}{m} \mid h \underset{\delta}{+} k) \end{vmatrix} = 0. \tag{3}$$

That is if all minors of order $(m + h)$ formed from the m columns $(n \mid \underset{\alpha}{m})$ and h others vanish, then all minors of order $(m + h + k)$ formed from the same m columns and $(h + k)$ others will vanish.

For

$$M' = \sum_{i=1}^{(m+h)_k} \begin{vmatrix} (n \mid m + \underset{\eta}{h} + k \mid \underset{i}{k}) \\ (\overline{n} \mid \underset{\alpha}{m} \mid h \underset{\delta}{+} k \mid \underset{\rho}{k}) \end{vmatrix} \cdot \begin{vmatrix} (n \mid \overline{m + \underset{\eta}{h} + k} \mid \underset{i}{k}) \\ (n \mid \underset{\alpha}{m})(\overline{n} \mid \underset{\alpha}{m} \mid \overline{h \underset{\delta}{+} k} \mid \underset{\rho}{k}) \end{vmatrix} = 0,$$

since the minors of order $(m + h)$ in every term are zero by hypothesis.

Since by (3) $M' = 0$, it is obvious that

$$\begin{vmatrix} (n \mid \underset{\alpha}{m} \mid \underset{j}{k})(n \mid m \underset{\epsilon}{+} h) \\ (n \mid \underset{\alpha}{m})(\overline{n} \mid \underset{\alpha}{m} \mid h \underset{\delta}{+} k) \end{vmatrix} = 0,$$

and we have

$$\sum_{i=1}^{m_k}\begin{vmatrix}(n \mid \underset{a}{m} \mid \underset{j}{k}) \\ (n \mid \underset{a}{m} \mid \underset{i}{k})\end{vmatrix} \cdot \begin{vmatrix}(n \mid m \underset{\epsilon}{+} h) \\ (n \mid \underset{a}{\overline{m}} \mid \underset{i}{k})(\overline{n} \mid \underset{a}{m} \mid h \underset{\delta}{+} k)\end{vmatrix} = 0$$

for $k = 1, 2, \cdots m$, and $j = 1, 2, \cdots m_k$.

The determinant of the coefficients in this set of equations is the kth compound of M and is therefore not zero.

It follows that

$$\begin{matrix}(n \mid m \underset{\epsilon}{+} h) \\ (n \mid \underset{a}{\overline{m}} \mid \underset{i}{k})(\overline{n} \mid \underset{a}{m} \mid h \underset{\delta}{+} k)\end{matrix}\;\Bigg| = 0 \;\Bigg| \tag{4}$$

for all values of ϵ, i, δ, k.

Giving k successively the values $1, 2, \cdots m$ we have our theorem proved, or stated symbolically it is:

If

$$\begin{vmatrix}(n \mid \underset{a}{m}) \\ (n \mid \underset{a}{m})\end{vmatrix} \neq 0, \qquad \text{and} \qquad \begin{vmatrix}(n \mid \underset{a}{m})(\overline{n} \mid \underset{a}{m} \mid \underset{\beta}{h}) \\ (n \mid \underset{a}{m})(\overline{n} \mid \underset{a}{m} \mid \underset{\gamma}{h})\end{vmatrix} = 0,$$

for

$$\left.\begin{matrix}\beta \\ \gamma\end{matrix}\right\} = 1, 2, \cdots (n - m)_h,$$

then

$$\begin{vmatrix}(n \mid \underset{a}{\overline{m}} \mid \underset{i}{g})(\overline{n} \mid \underset{a}{m} \mid h \underset{\delta}{+} g) \\ (n \mid \underset{a}{\overline{m}} \mid \underset{j}{k})(\overline{n} \mid \underset{a}{m} \mid h \underset{\epsilon}{+} k)\end{vmatrix} = 0, \tag{5}$$

for

$$\left.\begin{matrix}g \\ k\end{matrix}\right\} = 0, 1, \cdots m, \qquad \left.\begin{matrix}i \\ j\end{matrix}\right\} = 1, 2, \cdots m_k, \qquad \left.\begin{matrix}\delta \\ \epsilon\end{matrix}\right\} = 1, 2, \cdots (n - m)_{h+k}.$$

2. In a symmetric determinant, if all coaxial minors of order $m + h$ and $m + h + 1$, which contain the coaxial minor M of order m, are zero; then all minors of order $m + h$, which contain M, will be zero.

That is if

$$M \equiv \begin{vmatrix}(n \mid \underset{a}{m}) \\ (n \mid \underset{a}{m})\end{vmatrix},$$

and if

$$\begin{vmatrix}(n \mid \underset{a}{m})(\overline{n} \mid \underset{a}{m} \mid \underset{\beta}{h}) \\ (n \mid \underset{a}{m})(\overline{n} \mid \underset{a}{m} \mid \underset{\beta}{h})\end{vmatrix} = 0$$

and

$$D \equiv \begin{vmatrix} (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{\gamma}{h + 1}) \\ (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{\gamma}{h + 1}) \end{vmatrix} = 0$$

for all values of β and γ; then

$$\begin{vmatrix} (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{i}{h}) \\ (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{j}{h}) \end{vmatrix} = 0 \tag{6}$$

for all values of i and j.

For

$$\begin{vmatrix} (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{i}{h}) \\ (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{i}{h}) \end{vmatrix} \cdot \begin{vmatrix} (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{j}{h}) \\ (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{j}{h}) \end{vmatrix} - \begin{vmatrix} (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{i}{h}) \\ (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{j}{h}) \end{vmatrix}^2$$

is a coaxial minor of order two of the reciprocal of D and is therefore zero, and since each factor in the first term is zero it follows that

$$\begin{vmatrix} (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{i}{h}) \\ (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{j}{h}) \end{vmatrix} = 0.$$

If (6) is true and $M \neq 0$, it follows from (5) that every minor of order $m + h$ of a symmetric determinant is zero.

3. In a skew-symmetric determinant if the coaxial minor

$$M \equiv \begin{vmatrix} (n \mid \underset{\alpha}{m}) \\ (n \mid \underset{\alpha}{m}) \end{vmatrix}$$

is of even order, and if the coaxial minors of even order

$$\begin{vmatrix} (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{\beta}{h}) \\ (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{\beta}{h}) \end{vmatrix} = 0,$$

for all values of β; then

$$\begin{vmatrix} (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{i}{h}) \\ (n \mid \underset{\alpha}{m})(\bar{n} \mid \underset{\alpha}{m} \mid \underset{j}{h}) \end{vmatrix} = 0 \tag{7}$$

for all values of i and j.

For since $m + h + 1$ is odd, every coaxial minor of that order is zero. It follows then as in article 2 that every minor of the $(m + h)$th order which contains M is zero.

If (7) is true, and if $M \neq 0$, it follows from (5) that every minor of order $m + h$ of a skew-symmetric determinant is zero.

For other conditions determining the rank of a symmetric and a skew-symmetric determinant see the author's paper* in the American Journal of Mathematics for 1894.

SYRACUSE UNIVERSITY,
November, 1913.

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* Metzler, Compound Determinants, Amer. Jour. Math., vol. 16, pp. 131–150.